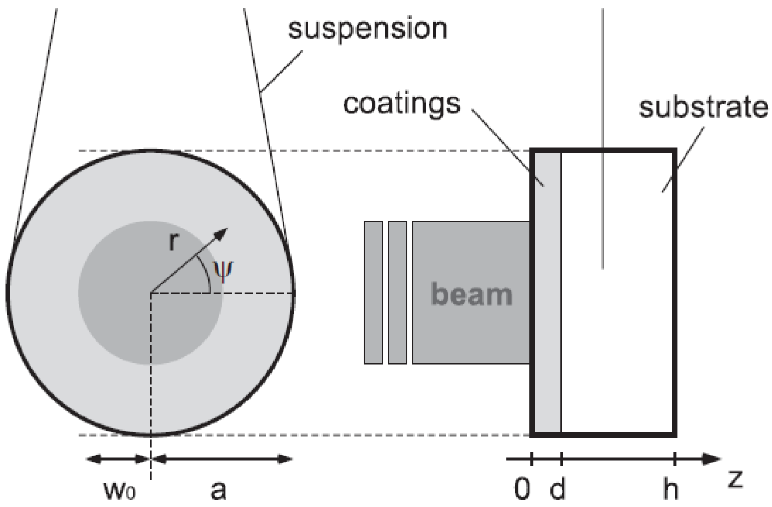


Substrate thermal noise

$$S_x = \frac{4k_BT}{\Omega}\frac{1-\nu^2}{\sqrt{\pi}Yw_0^2}\phi$$

Coating thermal noise

$$S_x = \frac{4k_BT}{\Omega}\frac{(1+\nu)(1-2\nu)2d_c}{\pi Yw_0^2}\phi_c$$

(Silica coating on Silica substrate; simple form)

How do we calculate the thermal-noise spectra?

"Fluctuation-dissipation Theorem (FdT)"

$$\frac{\text{"Internal energy of mass"}}{\text{"Thermal-noise kinetic energy"}} = \frac{\text{"Work done by external force"}}{\text{"Dissipated energy } W\text{"}}$$

$\Longrightarrow$ $$S_x = \frac{8k_BTW}{\Omega^2F_0^2}$$

$$W = U\Omega\phi$$

U: elastic energy, ϕ: loss

$$U = \frac{1}{2}\int \sum_{i,j} E_{ij} T_{ij} dV$$

(Strain tensor: E_{ij}; Stress tensor: T_{ij})

[strain tensor]

$$E_{rr} = \frac{\partial u_r}{\partial r}, \qquad E_{\psi\psi} = \frac{u_r}{r}, \qquad E_{zz} = \frac{\partial u_z}{\partial z},$$

$$E_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),$$

Lame coefficient

$$\lambda = \frac{Y\nu}{(1+\nu)(1-2\nu)},$$

$$\mu = \frac{Y}{2(1+\nu)}$$

(Y: Young's modulus, ν: Poisson ratio)

[stress tensor]

$$T_{rr} = (\lambda + 2\mu)E_{rr} + \lambda(E_{\psi\psi} + E_{zz}),$$

$$T_{\psi\psi} = (\lambda + 2\mu)E_{\psi\psi} + \lambda(E_{zz} + E_{rr}),$$

$$T_{zz} = (\lambda + 2\mu)E_{zz} + \lambda(E_{rr} + E_{\psi\psi}),$$

$$T_{rz} = 2\mu E_{rz}.$$

Hook's law + Newton's law ⇨ Elastic equation

$$2(1-\nu)\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) + (1-2\nu)\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial r} = 0, \quad \text{... (1)}$$

$$(1-2\nu)\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + 2(1-\nu)\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_r}{\partial z \partial r} + \frac{1}{r}\frac{\partial u_r}{\partial z} = 0, \quad \text{... (2)}$$

Roughly speaking, elastic eq can be written in the following form:
(here we assume $\partial u_z / \partial r = \partial u_r / \partial z$ and $u_j \rightarrow A_j(r) B_j(z)$)

$$(1) \rightarrow r^2 \frac{\partial^2 A_r}{\partial r^2} B_r + r \frac{\partial A_r}{\partial r} B_r - A_r B_r + r^2 A_r \frac{\partial B_r^2}{\partial z^2} = 0$$

$$\rightarrow \begin{cases} \dfrac{\partial^2 B_r}{\partial z^2} = f_c B_r & \text{second-order ODE} \\ r^2 \dfrac{\partial^2 A_r}{\partial r^2} + r \dfrac{\partial A_r}{\partial r} - A_r + r^2 A_r f_c = 0 & \text{Bessel equation} \end{cases}$$

With these and the boundary conditions:

$$T_{zz}(z=0) = -F_0 p(r), T_{rz}(z=0) = T_{rr}(r=\infty) = T_{rz}(r=\infty) = T_{zz}(z=\infty) = T_{rz}(z=\infty) = 0$$

the elastic eq can be solved as

$$\left(p(r) = \frac{2}{\pi w_0^2} e^{-2r^2/w_0^2} \right)$$

$$\begin{cases} u_r = \displaystyle\int_0^\infty \frac{e^{-k^2 w_0^2/8}}{4\pi\mu k} F_0 \left(-\frac{\mu}{\lambda+\mu} + kz \right) e^{-kz} J_1(kr) k dk \\ u_z = \displaystyle\int_0^\infty \frac{e^{-k^2 w_0^2/8}}{4\pi\mu k} F_0 \left(\frac{\lambda+2\mu}{\lambda+\mu} + kz \right) e^{-kz} J_0(kr) k dk \end{cases}$$

Using those and integrating $E_{ij}T_{ij}$ in the entire half-infinite plane, we get

$$U = \frac{1-\nu^2}{2\sqrt{\pi}Yw_0}F_0^2 \quad \text{(elastic energy to give substrate TN)}$$

If we integrate $E_{ij}T_{ij}$ in the coating (0<z<d), we get

$$U = \frac{(1+\nu)(1-2\nu)}{\pi Y w_0^2} d_c F_0^2 \quad \text{(elastic energy to give coating TN)}$$

This coating TN is with a same material for substrate and coatings.
If we want to calculate TN with tantala coating on silica substrate, we need strain/stress tensors inside the coatings.

$$E'_{rr} = E_{rr}, \qquad E'_{\psi\psi} = E_{\psi\psi}, \qquad E'_{rz} = E_{rz}, \qquad T'_{zz} = T_{zz}, \qquad T'_{rz} = T_{rz},$$

and also T'_{rr} ,$T'_{\psi\psi}$ and E'_{zz} from the following equations:

$$\begin{cases} T'_{rr} = (\lambda' + 2\mu')E'_{rr} + \lambda'(E'_{\psi\psi} + E'_{zz}), \\ T'_{\psi\psi} = (\lambda' + 2\mu')E'_{\psi\psi} + \lambda'(E'_{zz} + E'_{rr}), \\ T'_{zz} = (\lambda' + 2\mu')E'_{zz} + \lambda'(E'_{rr} + E'_{\psi\psi}), \end{cases}$$

Finally we get: $S_x(\Omega) = \frac{4k_B T}{\Omega}\frac{d}{\pi w_0^2}\frac{Y_c^2(1+\nu_s)^2(1-2\nu_s)^2 + Y_s^2(1+\nu_c)^2(1-2\nu_c)}{Y_s^2 Y_c(1-\nu_c^2)}\phi_c.$

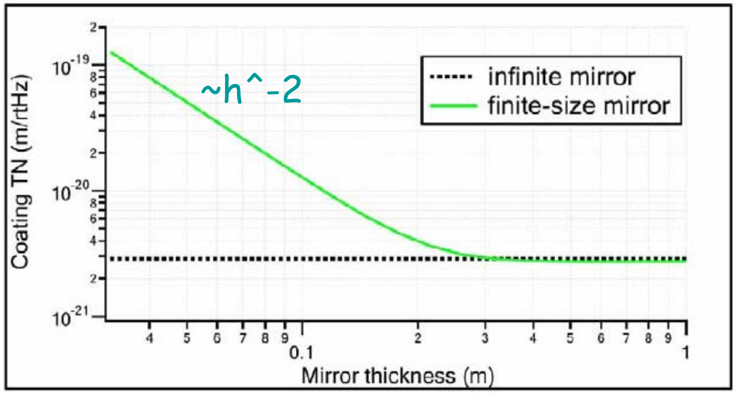
Coating TN

$$S_x(\Omega) = \frac{4k_B T}{\Omega}\frac{d}{\pi w_0^2}\frac{Y_c^2(1+\nu_s)^2(1-2\nu_s)^2 + Y_s^2(1+\nu_c)^2(1-2\nu_c)}{Y_s^2 Y_c(1-\nu_c^2)}\phi_c.$$

- TN decreases with small d_c ← large n_c or blue beam
- TN decreases with large w_0
- TN decreases with large Y_s ← Sapphire is better than Silicon
- TN decreases with small ϕ_c
- TN decreases with low T

Moreover, coating TN depends on mirror thickness.

Ex) ET mirror [mirror radius=31cm]



Mirror thickness h should not be smaller than mirror radius a.

[Thermoelastic noise and Thermorefractive noise]

Heat equation:

$$\dot{\theta}_j(z,t) - \kappa_j \nabla^2 \theta(z,t) = q_j(z,t)$$

W~(dθ/dz)²

temperature fluctuation

heat source

Heat source for TE noise: $$q_j^{TE} = -i\Omega \frac{\alpha_j Y_j T}{C_j(1-2\nu_j)} \Theta_j$$

$$\Theta = E_{rr} + E_{\psi\psi} + E_{zz}$$ (expansion)

Heat source for TR noise: $$q_c^{TR} = -i\Omega \frac{\beta_{eff} \tilde{\lambda} T F_0}{C_c} p(r)\delta(z)$$

While TE and TR should be calculated at once (thermo-optic noise), Substrate TE noise equation for example is as follows:

$$S_x = \frac{16 k_B T^2 (1+\nu)^2 \alpha_s^2 \kappa_s}{\sqrt{\pi} C_s w_0^3 \Omega^2}$$

It is proportional to T^2 instead of T.